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New exact solutions for polynomial oscillators in large dimensions

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Abstract

A new type of exact solvability is reported. The Schrödinger equation is considered in a very large spatial dimension $D \gg 1$ and its central polynomial potential is allowed to depend on 'many' $(= 2q)$ coupling constants. In a search for its bound states possessing an exact and elementary wavefunction ψ (proportional to a harmonic-oscillator-like polynomial of a freely varying, i.e., not just small, degree N), the 'solvability conditions' are known to form a complicated nonlinear set which requires a purely numerical treatment at a generic choice of D, q and N. Assuming that D is large we discovered and demonstrate that this problem may be *completely* factorized and acquires an amazingly simple exact solution at all N and up to $q = 5$ at least.

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1. Introduction

The key motivation of our present study of polynomial oscillators lies in the well-known fact that the *majority* of quantitative predictions in nuclear, atomic, molecular and condensed matter physics must rely on a more or less purely numerical model. Completely non-numerically tractable quantum systems are rare though, at the same time, useful and transparent (cf, e.g., the description of vibrations in molecules mimicked by the harmonic oscillator). Polynomial oscillators may be taken, in this setting, as lying somewhere on the borderline between the two regimes.

The first indications of a breakdown of the traditional separation between the numerical and analytic models in quantum mechanics came with the emergence of certain 'incompletely analytic' (now we call them quasi-exactly solvable, QES) polynomial oscillator models. Unfortunately, their separate discoveries at the beginning of the last quarter of the twentieth century [1–5] have all been understood and accepted as a mere formal curiosity. As an interesting confirmation of analogies with classical mechanics for the charged and shifted oscillator in two and three dimensions [1], as a peculiar singularity in a universal continued-fraction algorithm for sextic oscillators [2], as an exceptional case in the standard infinite-series solution of the corresponding differential Schrödinger equations [3–5], etc.

The defining property of the QES models (being solvable just for a part of the complete set of their coupling constants and/or energies) re-acquired a new meaning only after their Lie-algebraic re-interpretation [6] which revealed their genuine mathematical closeness to the exactly solvable (ES) Hamiltonians [7]. Subsequently, in physics, much of their relationship to the apparently different models has been revealed in a way summarized, e.g., in Ushveridze's detailed monograph [8].

In spite of the unique success of the *mathematical* QES formulae, a number of difficulties remained connected with their *practical* applications and applicability. One of the key reasons (and differences from the current ES models) is that the *explicit construction* of the multiplet QES energies *remains purely numerical*. Indeed, these values must be computed as roots of an N-dimensional secular determinant so that the difference between the variational, 'generic $N = \infty$' rule in Hilbert space seems only marginally simplified by the QES construction of any QES multiplet with $N \gg 1$.

The main purpose of our present study is related precisely to the latter point. Our idea may be explained, briefly, as an application of perturbative philosophy assuming that the *spatial* dimension D is large. In this spirit, we are going to consider a general Magyari [4] QES Hamiltonian $H^{(q,N)}(D)$ (at a fixed dimension D of the space of coordinates and with a chosen size N of its secular determinants, see below for a more detailed explanation). Finally, we *construct a set* of its specific approximations $H_0^{(q,N)}(\infty)$ with errors proportional to $1/D$.

The *exact solvability* of the latter Hamiltonians $H_0^{(q,N)}$ emerged as an utterly unexpected result of our (originally, fully numerical) calculations. Our presentation starts from a concise review of the concepts of exact solvability in section 2. Section 3 will then mention a few specific formal merits of transition to the domain of large dimensions D which simplifies the general Magyari secular determinants considerably. The core of our message appears in section 4 where in the domain of $D \gg 1$, the separate families of the polynomial oscillators (numbered by the integer $q = 1, 2$ and 3) are studied in detail and shown to lead to the closed solutions. Section 5 outlines the possibilities of an extension of these results to $q = 4$ and 5 while section 6 summarizes and discusses some possible practical consequences.

2. Exactly solvable oscillators: a brief review

2.1. Harmonic oscillator and the like: all bound states are elementary

One of the most exceptional exactly solvable models in quantum mechanics is the central harmonic oscillator in D dimensions. Its so-called superintegrability (the term coined by Winternitz [9]) makes its partial differential Schrödinger equation

$$\left(-\frac{\hbar^2}{2m}\triangle + \frac{1}{2}m\Omega^2|\vec{x}|^2\right)\Psi(\vec{x}) = \varepsilon\Psi(\vec{x}) \tag{1}$$

solvable by the separation of variables in several systems of coordinates. The most common Cartesian choice may be recommended for the first few lowest spatial dimensions D only [10]. In contrast, the separation in the spherical system remains equally transparent at any D because it reduces equation (1) to *the same* ordinary (so-called radial) differential equation

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\ell(\ell+1)}{r^2} + \omega^2 r^2\right]\psi(r) = E\psi(r) \tag{2}$$

with $r = |\vec{x}| \in (0, \infty)$, $E = 2m\varepsilon/\hbar^2$ and $\omega = m\Omega/\hbar > 0$. In this language, we have $\ell = \ell_L = L + (D-3)/2$ where $L = 0, 1, \ldots$ [11]. At each L the energy levels are numbered by the second integer,

$$E = E_{n,L} = \omega(2n + \ell_L + 3/2) \qquad n, L = 0, 1, \ldots. \tag{3}$$

The wavefunctions with quadratic $\lambda(r) = \omega r^2/2 > 0$ and minimal $N = n + 1$ in

$$\psi_{n,L}(r) = r^{\ell+1}\,\mathrm{e}^{-\lambda(r)} \sum_{m=0}^{N-1} h_m r^{2m} \tag{4}$$

are proportional to an nth Laguerre polynomial [12]. In Hilbert space, their set is complete: this may be explained via oscillation theorems and characterizes the harmonic oscillator as exceptional. In such a setting, one should recollect all the similar (i.e., Coulomb and Morse) exactly solvable potentials, but one need not mention them separately as long as they are formally equivalent to our harmonic oscillator example after a simple change of variables [13].

2.2. *Sextic oscillators: many bound states are elementary*

For many phenomenological purposes, even the shifted harmonic oscillator forces $V^{(\mathrm{HO})}(r) = V^{(\mathrm{HO})}(0) + \omega^2 r^2$ are not flexible enough. Fortunately, there exists an immediate QES generalization of the harmonic oscillators revealed by Singh *et al* [2]. Let us summarize briefly this type of solvability as a construction which starts from a replacement of the free constants in $V^{(\mathrm{HO})}(r)$ by their polynomial descendants of the first order in r^2,

$$\omega \longrightarrow W(r) = \alpha_0 + \alpha_1 r^2 \qquad V^{(\mathrm{HO})}(0) \longrightarrow G_{-1} + G_0 r^2 = U(r). \tag{5}$$

After the conventional choice of $G_{-1} = 0$, this trick gives the general sextic potential

$$V^{(\mathrm{sextic})}(r) = U(r) + r^2 W^2(r) = g_0 r^2 + g_1 r^4 + g_2 r^6 \tag{6}$$

all three couplings of which are simple functions of our initial three parameters and vice versa,

$$g_2 = \alpha_1^2 > 0 \qquad g_1 = 2\alpha_0\alpha_1 \qquad g_0 = 2\alpha_0^2 + G_0. \tag{7}$$

The resulting Schrödinger bound state problem cannot be solved in closed form. Nevertheless, we may postulate the polynomiality of the wavefunctions $\psi_{n,L}^{(\mathrm{sextic})}(r)$ *for a finite multiplet* (i.e., *N*-plet) of the wavefunctions. For this purpose, it is necessary to lower the number of the freely variable couplings by the specific constraint

$$G_0 = -\alpha_0^2 - \alpha_1(4N + 2\ell + 1) \qquad N \geqslant 1. \tag{8}$$

For each *N*-plet, the polynomial solutions (4) are made exact by the N-dependent QES condition (8). The choice of a WKB-like (i.e., quartic) exponent

$$\lambda(r) = \tfrac{1}{2}\alpha_0 r^2 + \tfrac{1}{4}\alpha_1 r^4 \tag{9}$$

guarantees their physical normalizability. The ansatz (4) then transforms the differential Schrödinger equation into a linear algebraic definition of the unknown *N*-plet of coefficients h_m. An incomplete solution is always obtained for a mere finite set of the levels $n \in (n_0, n_1, \ldots, n_{N-1})$. In contrast to the harmonic oscillator, the QES solvability is based on the *L*- and *N*-dependent constraints (8) so that, generically, the elementary QES multiplet exists in a single partial wave only.

2.3. General harmonic-oscillator-like bound states

The explicit energy formula (3) for the harmonic oscillator was replaced by an implicit definition in the preceding paragraph which gives the sextic QES energies *in the purely numerical form*, namely as zeros of Singh's secular determinant of a certain tridiagonal *N* by *N* matrix [8]. In this sense, a very natural further extension of Singh's QES construction exists and has been described by Magyari [4]. Its description may start from the more consequent replacement (5), constructed from even polynomials of degree $2q$,

$$\begin{aligned} &V^{(q)}(r) = U^{(q)}(r) + r^2[W^{(q)}(r)]^2 \qquad U^{(q)}(r) = G_0 r^2 + G_1 r^4 + \cdots + G_{q-1} r^{2q} \\ &W^{(q)}(r) = \alpha_0 + \alpha_1 r^2 + \cdots + \alpha_q r^{2q}. \end{aligned} \tag{10}$$

This formula re-parametrizes the polynomial

$$V^{[q]}(r) = g_0 r^2 + g_1 r^4 + \cdots + g_{2q} r^{4q+2} \qquad g_{2q} = \gamma^2 > 0 \tag{11}$$

and specifies the one-to-one correspondence between the two sets of couplings,

$$\{g_0, \ldots, g_{2q}\} \Longleftrightarrow \{G_0, \ldots, G_{q-1}, \alpha_0, \ldots, \alpha_q\}$$

where $g_{2q} = {\alpha_q}^2$, $g_{2q-1} = g_{2q-1}(\alpha_q, \alpha_{q-1}) = 2\alpha_{q-1}\alpha_q, \ldots$ or, in the opposite direction, $\alpha_q = \sqrt{g_{2q}} \equiv \gamma > 0$, $\alpha_{q-1} = g_{2q-1}/(2\alpha_q)$, etc.

At a generic $q = 1, 2, \ldots$, equation (9) must be further modified in such a way that $rW(r) \equiv Z'(r)$,

$$\lambda^{(q)}(r) = \frac{1}{2}\alpha_0 r^2 + \frac{1}{4}\alpha_1 r^4 + \cdots + \frac{1}{2q+2}\alpha_q r^{2q+2}. \tag{12}$$

With $\alpha_q > 0$, one verifies that

$$\psi^{(\text{physical})}(r) \approx \mathrm{e}^{-\lambda^{(q)}(r)+\mathcal{O}(1)} \qquad r \gg 1$$

which means that the correct bound-state ansatz

$$\psi(r) = \sum_{n=0}^{N-1} h_n^{(N)} r^{2n+\ell+1} \exp[-\lambda^{(q)}(r)] \tag{13}$$

converts our radial equations (2) and (11) into an equivalent linear algebraic problem

$$\hat{Q}^{[N]}\vec{h}^{(N)} = 0. \tag{14}$$

Closer inspection reveals that this problem is overcomplete, i.e., its matrix is *non-square* and asymmetric,

$$\hat{Q}^{[N]} = \begin{pmatrix} B_0 & C_0 & & & & & \\ A_1^{(1)} & B_1 & C_1 & & & & \\ \vdots & & \ddots & \ddots & & & \\ A_q^{(q)} & \ldots & A_q^{(1)} & B_q & C_q & & \\ & \ddots & & & \ddots & \ddots & \\ & & A_{N-2}^{(q)} & \ldots & A_{N-2}^{(1)} & B_{N-2} & C_{N-2} \\ & & & A_{N-1}^{(q)} & \ldots & A_{N-1}^{(1)} & B_{N-1} \\ & & & & \ddots & \vdots & \vdots \\ & & & & & A_{N+q-2}^{(q)} & A_{N+q-2}^{(q-1)} \\ & & & & & & A_{N+q-1}^{(q)} \end{pmatrix}. \tag{15}$$

Its elements depend on the parameters in a bilinear manner,

$$\begin{aligned} &C_n = (2n+2)(2n+2\ell+3) && B_n = E - \alpha_0(4n+2\ell+3) \\ &A_n^{(1)} = -\alpha_1(4n+2\ell+1) + \alpha_0^2 - g_0, && A_n^{(2)} = -\alpha_2(4n+2\ell-1) + 2\alpha_0\alpha_1 - g_1, \\ &\qquad \ldots \\ &A_n^{(q)} = -\alpha_q(4n+2\ell+3-2q) + (\alpha_0\alpha_{q-1} + \alpha_1\alpha_{q-2} + \cdots + \alpha_{q-1}\alpha_0) - g_{q-1} \\ &\qquad\qquad n = 0, 1, \ldots . \end{aligned} \tag{16}$$

At *any* fixed and finite dimension $N = 1, 2, \ldots,$ the non-square system (14) is an over-determined set of $N + q$ linear equations for the N non-vanishing components of the vector $\vec{h}^{(N)}$.

2.4. Changes of variables and an extension of applicability of the harmonic-oscillator-like constructions

At $q = 0$, equations (15) degenerate back to the recurrences and define the well-known harmonic oscillator states. As already mentioned, their additional merit lies in the coincidence of their polynomial part with the current Laguerre polynomials.

At $q = 1$, we return to the sextic model where the 'redundant' last row fixes one of the couplings and where we are left with a diagonalization of an N by N matrix which defines the N-plet of the real QES energies in principle. In such a setting, an important piece of additional encouragement results from the well-known possibility of a definition of new QES Hamiltonians by a change of variables $r \to x$ and $\psi(r) \to x^{\text{const}}\phi(x)$ in the Schrödinger equation [13, 14]. Even when considered just in the most elementary power-law form, this change is defined by the prescription [15]

$$r^{2j} \longrightarrow x^{\delta(k)} \qquad \delta(k) = 2\frac{j+1}{k} - 2 \qquad k = 1, 2, \ldots, 2q+2$$

and extends the class of 'interesting' potentials not only in the well-known $q = 0$ case (see the open possibility of a transition to the completely solvable Coulombic potential as mentioned above) but also in the $q = 1$ model where the following four equivalent potentials may be distinguished and numbered by the above index k attached to them as their second superscript,

$$\begin{aligned} V^{(q=1,k=1)}(r) &= ar^2 + br^4 + r^6 \\ V^{(q=1,k=2)}(r) &= ar^{-1} + br + r^2 \\ V^{(q=1,k=3)}(r) &= ar^{-4/3} + br^{-2/3} + r^{2/3} \\ V^{(q=1,k=4)}(r) &= ar^{-3/2} + br^{-1} + cr^{-1/2}. \end{aligned} \tag{17}$$

The situation is more complicated at $q > 1$. The counting of parameters and equations indicates that unless one broadens the class of potentials, only a very small multiplet of bound states may remain available in closed form [14]. Still, the same elementary change of variables enables us to extend the set of partially solvable potentials to the six-member family at $q = 2$,

$$\begin{aligned} V^{(q=2,k=1)}(r) &= ar^2 + br^4 + cr^6 + dr^8 + r^{10} \\ V^{(q=2,k=2)}(r) &= ar^{-1} + br + cr^2 + dr^3 + r^4 \\ V^{(q=2,k=3)}(r) &= ar^{-4/3} + br^{-2/3} + cr^{2/3} + dr^{4/3} + r^2 \\ V^{(q=2,k=4)}(r) &= ar^{-3/2} + br^{-1} + cr^{-1/2} + dr^{1/2} + r \end{aligned}$$

$$V^{(q=2,k=5)}(r) = ar^{-8/5} + br^{-6/5} + cr^{-4/5} + dr^{-2/5} + r^{2/5}$$
$$V^{(q=2,k=6)}(r) = ar^{-5/3} + br^{-4/3} + cr^{-1} + dr^{-2/3} + fr^{-1/3} \tag{18}$$

etc (cf, e.g., [15] where similar lists of the partially solvable potentials have been displayed up to $q = 4$). In this way, the availability of exact solutions for all these various forces might offer a new inspiration, say, in some phenomenological considerations and models and/or for their perturbative analyses and some more detailed large-D calculations.

3. Solvability of polynomial oscillators at large spatial dimensions

Up to now, our attention has been concentrated upon the structure of the QES wavefunctions. From the point of view of the evaluation of the energies, the main dividing line between the solvable and unsolvable spectra is in fact marked by the distinction between the closed $q = 0$ formulae and their implicit QES form at $q = 1$. The transition to the next $q = 2$ may be perceived as merely technical. At all $q \geqslant 1$, the difficulties grow with N. In such a setting we noticed the emergence of certain simplifications at $D \gg 1$.

3.1. Difficulties arising at $q \geqslant 1$

At any D, the last row in equation (14) decouples from the rest of the system. At any $q > 1$ it may be treated as a constraint which generalizes equation (8),

$$g_{q-1} = -\alpha_q(4n + 2\ell + 3 - 2q) + (\alpha_0\alpha_{q-1} + \alpha_1\alpha_{q-2} + \cdots + \alpha_{q-1}\alpha_0). \tag{19}$$

The insertion of this explicit definition of the coupling g_{q-1} simplifies the lowest diagonal in $\hat{Q}^{[N]}$,

$$A_n^{(q)} = 4\gamma(N + q - n - 1). \tag{20}$$

Since $A_{N+q-1}^{(q)} = 0$, we may omit the last line from equation (15) and drop the 'hat' ($\hat{}$) of the diminished matrix $\hat{Q}^{[N]}$. This gives equation (14) in the more compact form

$$Q^{[N]}\vec{h}^{(N)} = 0 \tag{21}$$

where the size of the non-square matrix $Q^{[N]}$ is merely $(N + q - 1)$ by N. Unfortunately, the new equation is still purely numerical, with the exception of the simplest special case with $q = 0$ where no coupling is fixed and where the energies themselves are given by the explicit formula (19). At $q = 0$ also the recurrences for coefficients of the wavefunctions may be solved in a compact form.

As already mentioned, the $q = 1$ version of equation (21) degenerates to the secular equation

$$\det Q^{[N]} = 0. \tag{22}$$

This is a purely numerical problem at all the larger $N \geqslant 5$. Still, one coupling is fixed by equation (19) and only the N-plet of energies must be calculated as the real zeros of a secular polynomial.

At $q \geqslant 2$, the q independent (and mutually coupled) N by N secular determinants must vanish at once [16]. With an auxiliary abbreviation for the energy $E = -g_{-1}$ this means that at least one of the couplings is always energy-dependent and its value must be determined numerically. In the other words, our non-square matrix $Q^{[N]} = Q^{[N]}(g_{-1}, g_0, \ldots, g_{q-2})$ will annihilate the vector $\vec{h}^{(N)}$ if and only if all its q arguments are determined in a deeply nonlinear and self-consistent, mostly purely numerical manner.

3.2. Simplifications arising at $D \gg 1$

For a clear understanding of what happens at $D \gg 1$, let us pick up the $q = 0$ model $V^{[q]}(r)$ and re-consider its coordinate-dependence in the $D \geqslant 1$ regime. We discover a quick growth of the minimum of the effective potential, occurring at a fairly large value of the coordinate $R = R(D) = [\ell(\ell+1)/\omega^2]^{1/4} \gg 1$. Its Taylor expansion

$$\frac{\ell(\ell+1)}{r^2} + \omega^2 r^2 = 2\omega^2 R^2 + 4\omega^2 (r-R)^2 - \frac{4}{R}(r-R)^3 + \cdots \tag{23}$$

reveals that the shape of the effective potential is R- and D-independent. Near the minimum, also the cubic and higher corrections become negligible. This implies that the shifted harmonic oscillations now characterize the local solutions very well. In particular, we have the wavefunctions

$$\psi_0 \sim \mathrm{e}^{-\omega(r-R)^2}, \quad \psi_1 \sim (r-R)\,\mathrm{e}^{-\omega(r-R)^2}, \ \ldots \qquad r \approx R \tag{24}$$

and re-derive also the leading-order degeneracy of the spectrum and its equidistance in the next order,

$$E_0 = 2\omega^2 R^2 + 2\omega + \cdots, \qquad E_1 = 2\omega^2 R^2 + 6\omega + \cdots, \ldots. \tag{25}$$

The agreement of this approximate formula with the available exact spectrum (3) is amazing.

Considerations which have led to this agreement may be applied as a guide to the large-D description of the low-lying states in *any* phenomenological input potential well $V(r)$. In such a context, a skeptical question is due. Once we have the exact formula, why should we search for its alternative (re-)derivation? The reply will follow from our forthcoming results. We shall see that a qualitatively different new source of $D \gg 1$ simplification will emerge in *all* the exactly solvable polynomial $q < \infty$ models.

3.3. Magyari equations in the large-D regime

In the above $D \gg 1$ construction, little information can be extracted from the wavefunctions themselves. Although we Taylor-expanded the effective potential, we did not make any use of the information about the wavefunctions. In this sense, we are now going to demonstrate the feasibility of the approach where the *guaranteed* polynomiality of the wavefunctions will play a key role.

In our original differential equation (2) as well as in all its $q > 0$ generalizations, the numerical value of the spatial dimension D will be assumed large. This will simplify our matrix re-arrangement (21) of this problem with the matrix elements (16) re-written as *linear* functions of D,

$$\begin{aligned} &C_n = (2n+2)(2n+2L+D) \qquad B_n = -g_{-1} - \alpha_0(4n+2L+D) \\ &A_n^{(k)} = -g_{k-1} - \alpha_k(4n+2L+D-2k) + (\alpha_0\alpha_{k-1} + \cdots + \alpha_{k-1}\alpha_0) \\ &\qquad k = 1, 2, \ldots, q-1 \qquad n = 0, 1, \ldots, N+q-2. \end{aligned} \tag{26}$$

Besides the D-independent $A_n^{(q)} = A_n^{(q)[0]}$ which remains unchanged, we shall preserve the dominant components of the matrix elements,

$$C_n^{[0]} = (2n+2)D \qquad B_n^{[0]} = -g_{-1} - \alpha_0 D \qquad A_n^{(k)[0]} = -g_{k-1} - \alpha_k D \qquad k < q.$$

Then we re-scale the coordinates and, hence, coefficients,

$$h_n^{(N)} = p_n/\mu^n. \tag{27}$$

By the choice of the parameter μ, we are free to achieve that the uppermost and the lowest diagonals are just a re-ordering of each other,

$$\frac{2D}{\mu} = \tau = 4\gamma\mu^q. \tag{28}$$

This exhausts the freedom and fixes the D-dependence of our scaling,

$$\mu = \mu(D) = \left(\frac{D}{2\gamma}\right)^{1/(q+1)} \qquad \tau = \tau(D) = (2^{q+2}D^q\gamma)^{1/(q+1)}. \tag{29}$$

The recipe replaces the energies and couplings $\{g_{-1}, g_0, \ldots, g_{q-2}\}$ by the new parameters $\{s_1, s_2, \ldots, s_q\}$ in a linear way,

$$g_{k-2} = -\alpha_{k-1}D - \frac{\tau}{\mu^{k-1}}s_k \qquad k = 1, 2, \ldots, q. \tag{30}$$

In the leading-order approximation, our re-scaled Magyari equations read

$$\begin{pmatrix} s_1 & 1 & & & & \\ s_2 & s_1 & 2 & & & \\ \vdots & & \ddots & \ddots & & \\ s_q & \vdots & & s_1 & N-2 & \\ N-1 & s_q & & & s_1 & N-1 \\ & N-2 & s_q & & \vdots & s_1 \\ & & \ddots & \ddots & & \vdots \\ & & & 2 & s_q & s_{q-1} \\ & & & & 1 & s_q \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{N-2} \\ p_{N-1} \end{pmatrix} = 0. \tag{31}$$

We shall now study their solutions.

3.4. *Inspiration: new closed formulae at* $q = 0$

At $q = 0$, as already mentioned, the energies are unique functions of N and do not exhibit any unexpected behaviour. Still, it is instructive to extract the leading-order result for wavefunctions. The $q = 0$ version of equation (31)

$$\begin{pmatrix} N-1 & 1 & & & & \\ & N-2 & 2 & & & \\ & & \ddots & \ddots & & \\ & & & 2 & N-2 & \\ & & & & 1 & N-1 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{N-2} \\ p_{N-1} \end{pmatrix} = 0 \tag{32}$$

defines the (up to the normalization, unique) Taylor coefficients

$$p_n = (-1)^n p_0 \binom{N-1}{n}. \tag{33}$$

We may appreciate that they do not carry any round-off error and that the related leading-order wavefunctions possess the elementary form

$$\psi_{n,L}(r) = r^{\ell+1}\left(1 - \frac{r^2}{\mu}\right)^n \mathrm{e}^{-\lambda r^2} \qquad n = 0, 1, \ldots. \tag{34}$$

At the same time, one must be aware that only the leading-order part of (34) is to be compared with the available exact $D < \infty$ result. In particular, the presence of a degenerate nodal zero is an artefact of the zero-order construction. All this experience may serve as a guide to the less transparent $q > 0$ cases.

4. New partially solvable models with $q \geqslant 1$, any N and large $D \gg 1$

In the way inspired by equation (18), one may move beyond $q = 0$ and $q = 1$ and transform the decadic forces into their quartic equivalents etc. Paper [15] may be consulted for details which indicate that the study of any potential $V(r)$ *which is a polynomial in the powers of the coordinate* r may be replaced by the study of its present Magyari or 'canonical' QES representation $V^{(q)}(r)$ at a suitable integer q. In addition, we shall also restrict our attention to the domain of large D.

4.1. Guide: sextic QES oscillator with $q = 1$ and any N

Starting from the first nontrivial sextic-oscillator potential (6) with $q = 1$ and with the binding energies re-parametrized in accord with equation (30) where $s_1 = s$,

$$E = \frac{1}{2}\frac{g_1}{\sqrt{g_2}}D + (64g_2)^{1/4}\sqrt{D}s$$

full attention must be paid to the self-consistency problem represented by the set of equations (31). At every N, its first nontrivial $q = 1$ version

$$\begin{pmatrix} s & 1 & & & & \\ N-1 & s & 2 & & & \\ & N-2 & s & 3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 2 & s & N-1 \\ & & & & 1 & s \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{N-2} \\ p_{N-1} \end{pmatrix} = 0 \tag{35}$$

has the form of an asymmetric eigenvalue problem. In standard manner, it leads to the secular equation (22) expressible as the following sequence of polynomial conditions,

$$\begin{aligned} &s^3 - 4s = 0 && N = 3 \\ &s^4 - 10s^2 + 9 = 0 && N = 4 \\ &s^5 - 20s^3 + 64s = 0 && N = 5 \end{aligned}$$

etc. By mathematical induction, all the infinite hierarchy of these equations has been recently derived and solved in [17].

Quite remarkably, all of the *real* (i.e., 'physical') energy roots $s = s^{(j)}$ proved to be equal to integers. Moreover, all of them may be determined by the single and compact formula

$$s = s^{(j)} = -N - 1 + 2j \qquad j = 1, 2, \ldots, N. \tag{36}$$

This represented one of the key motivations of our present work, especially when we imagined that also all the related coefficients $p_n^{(j)}$ may equally easily be normalized to integers,

$$\begin{aligned} &p_0^{(1)} = 1 && N = 1 \\ &p_0^{(1)} = p_1^{(1)} = p_0^{(2)} = -p_1^{(2)} = 1 && N = 2 \end{aligned}$$

$$p_0^{(1)} = p_2^{(1)} = p_0^{(2)} = -p_2^{(2)} = p_0^{(3)} = p_2^{(3)} = 1$$

$$p_1^{(1)} = -p_1^{(3)} = 2 \qquad p_1^{(2)} = 0 \qquad N = 3$$

etc.

The first result of our subsequent computations using the symbolic manipulation techniques proved equally encouraging since we succeeded in compactification of the set of above recurrent solutions to the single leading-order form of the related wavefunctions,

$$\psi^{(j)}(r) = r^{\ell+1}\left(1+\frac{r^2}{\mu}\right)^{N-j}\left(1-\frac{r^2}{\mu}\right)^{j-1}\exp\left(-\frac{1}{2}\alpha_0 r^2 - \frac{1}{4}\alpha_1 r^4\right) \qquad j = 1, 2, \ldots, N. \tag{37}$$

A few more comments may be added.

- The large and degenerate nodal zeros in equation (37) are a mere artefact of the zero-order construction.
  (1) The apparently interesting exact summability of all the separate $\mathcal{O}(r^2/\mu)$ error terms is not too relevant, indeed. Although it leads to the zero-order nodes at $r = \mathcal{O}(\sqrt{\mu}) = \mathcal{O}(D^{1/4})$, these nodes have no real physical meaning.

- The leading-order perturbative approximation provides reliable information about the energies.
  (1) They are asymptotically degenerate, due to the large overall shift of the energy scale as explained in section 3.2.
  (2) The next-order corrections may be easily obtained by the recipes of the textbook perturbation theory.
  (3) As long as the coefficients p_n are defined in integer arithmetic, the latter strategy gives, by construction, all the above-mentioned energy corrections without any rounding errors in a way outlined in more detail in [17].

In the other words, we may say that formula (37) may either be truncated to its leading-order form $\psi^{(j)}(r) = r^{\ell+1}\exp(-\lambda^{(2)}(r))$ or, better, its full form may be used as a generating function which facilitates the explicit evaluation of the coefficients $p_n^{(j)}$. In comparison with the oversimplified harmonic oscillator, the $q = 1$ wavefunctions may be characterized by a similar coordinate dependence which becomes spurious (i.e., dependent on the selected normalization) everywhere beyond the perturbatively accessible domain of r.

At the same time, the energies specified by equation (36) form an absolutely amazing multiplet. On the background of its existence, a natural question arises whether some similar regularities could also emerge at some of the larger integer indices $q > 1$. We are now going to demonstrate that in spite of the growth of the technical obstacles in dealing with the corresponding key equation (31), the answer is, definitely, affirmative.

4.2. *The first generalization: decadic oscillators with $q = 2$ and any N*

The decadic anharmonic oscillator exhibits certain solvability features which motivated its deeper study in the non-Hermitian context [18]. The changes of variables make this oscillator very closely related to the common quartic problem with recognized relevance of both its non-Hermitian [19] and Hermitian [8, 16, 20] QES constructions.

Paying attention to the $D \gg 1$ domain and abbreviating the parameters $s_1 = s$ and $s_2 = t$ of the respective decadic-oscillator energy and coupling in equation (30), we arrive at the four-diagonal version of our solvability condition (31) at $q = 2$,

$$\begin{pmatrix} s & 1 & & & & & & \\ t & s & 2 & & & & & \\ N-1 & t & s & 3 & & & & \\ & N-2 & t & s & 4 & & & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & & & 3 & t & s & N-1 \\ & & & & 2 & t & s \\ & & & & & 1 & t \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{N-2} \\ p_{N-1} \end{pmatrix} = 0. \tag{38}$$

This is the first really nontrivial equation of the class (31). In order to understand its algebraic structure in more detail, let us first choose the trivial case with $N = 2$ and imagine that the resulting problem

$$\begin{pmatrix} s & 1 \\ t & s \\ 1 & t \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} = 0$$

(with $p_1 \neq 0$ due to the definition of N) may be solved by the determination of the unknown ratio of the wavefunction coefficients $p_0/p_1 = -t$ from the last line, and by the subsequent elimination of $t = 1/s$ using the first line. The insertion of these two quantities transforms the remaining middle line into the cubic algebraic equation $s^3 = 1$ with the single real root $s = 1$.

The next equation at $N = 3$ is still worth mentioning because it shows that the strategy accepted in the previous step is not optimal. Indeed, in

$$\begin{pmatrix} s & 1 & 0 \\ t & s & 2 \\ 2 & t & s \\ 0 & 1 & t \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix} = 0 \tag{39}$$

the same elimination of $p_1/p_2 = -t$ and of $p_0/p_2 = (t^2 - s)/2$ from the third line leads to the apparently ugly result

$$st^2 - s^2 - 2t = 0 \qquad t^3 - 3st + 4 = 0.$$

Still, one should not feel discouraged, at least for the following two reasons. Firstly, an alternative strategy starting from the elimination of p_0 and p_2 leads to the much more symmetric pair of conditions

$$t^2 - s^2 t + 2s = 0 \qquad s^2 - t^2 s + 2t = 0$$

the respective pre-multiplication of which by t and s gives the difference $t^3 = s^3$. This means that $t = \varepsilon s$ where the three eligible proportionality constants exist such that $\varepsilon^3 = 1$. Thus, our problem degenerates to a quadratic equation with the pair of real roots $s = t = s^{(1,2)}$ such that

$$s^{(1)} = 2 \qquad s^{(2)} = -1.$$

The second reason for optimism is even stronger: The 'ugliness' of the results of the elimination may be re-interpreted as an inessential intermediate stage of the solution of equation (39) for all its four unknowns (we may always put $p_N = 1$) by the 'brute-force' symbolic manipulations on the computer, by the so-called technique of Gröbner bases [21]. In particular, at $N = 3$, the computerized experiment of this type leads to the step-by-step elimination of the redundant unknowns and to the final effective polynomial equation for the single unknown quantity s,

$$s^6 - 7s^3 - 8 = 0.$$

This equation may be verified to possess the same complete set of real roots as above. One may conclude that the real energies of the 'strongly spiked' decadic oscillator are very easily determined even without a detailed specification of an 'optimal' elimination pattern.

We see that in general one may expect that equation (38) may give many unphysical complex roots as well. This is confirmed by the next step with $N = 4$ leading to the effective polynomial equation

$$s^{10} - 27s^7 + 27s^4 - 729s = 0.$$

Being tractable by standard computer software, it results in the set of the mere two real roots again,

$$s^{(1)} = 3 \qquad s^{(2)} = 0 \qquad N = 4 \quad q = 2.$$

It is not difficult to continue along the same path. In general one finds that the $q = 2$ problem may be reduced to a single polynomial equation with $\binom{N+1}{2}$ complex roots s. Still, originally, we were unable to suspect that after all the explicit calculations, *all* the general physical (i.e., real) spectrum of energies proves to be quite rich and appears described again by the following *closed and still almost trivial* formula

$$s^{(j)} = N + 2 - 3j \qquad j = 1, 2, \ldots, j_{\max} \quad j_{\max} = \text{entier}\left[\frac{N+1}{2}\right]. \tag{40}$$

This is our first important conclusion. After one applies the same procedure at higher and higher dimensions N, some more advanced symbolic manipulation tricks must be used [21]. Nevertheless, one repeatedly arrives at the confirmation of the N-independent empirical observation (40) and extends it by another rule that at all the values of the dimension N, there exist only such real roots that $s^{(j)} = t^{(j)}$. This means that each 'solvability admitting' real energy s requires, purely constructively, the choice of its own 'solvability admitting' real coupling constant t.

This is our second important result which parallels completely the similar observations made in our preceding paper on the quartic oscillators [20]. Now, a fully open question arises in connection with all the $q > 2$ versions of equation (31). Do their real roots $s, t, \ldots$ exhibit a similar pattern as emerged at $q = 2$?

4.3. The second generalization: oscillators with $q = 3$ and their solvability at any N

At $q = 3$, we have to solve the five-diagonal equation (31),

$$\begin{pmatrix} r & 1 & & & & & & & \\ s & r & 2 & & & & & & \\ t & s & r & 3 & & & & & \\ N-1 & t & s & r & 4 & & & & \\ & N-2 & t & s & r & 5 & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & 4 & t & s & r & N-1 \\ & & & & 3 & t & s & r \\ & & & & & 2 & t & s \\ & & & & & & 1 & t \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{N-2} \\ p_{N-1} \end{pmatrix} = 0 \tag{41}$$

which may be reduced, by means of similar symbolic computations as above, to the single polynomial problem

$$t^9 - 12t^5 - 64t = 0$$

at $N = 3$, to the similar condition

$$t^{16} - 68t^{12} - 442t^8 - 50116t^4 + 50\,625 = 0$$

at $N = 4$, to the conditions of vanishing of the secular polynomial

$$t^{25} - 260t^{21} + 7280t^{17} - 1039\,040t^{13} - 152\,089\,600t^9 + 2030\,239\,744t^5 + 10\,485\,760\,000t$$

at $N = 5$, or to the perceivably longer equation

$$\begin{aligned} t^{36} - 777t^{32} + 135\,716t^{28} - 17\,189\,460t^{24} - 3513\,570\,690t^{20} \\ - 1198\,527\,160\,446t^{16} + 103\,857\,100\,871\,252t^{12} + 873\,415\,814\,269\,404t^8 \\ + 74\,500\,845\,455\,535\,625t^4 - 75\,476\,916\,312\,890\,625 = 0 \end{aligned}$$

at $N = 6$, etc. These computations represent a difficult technical task but at the end they reveal again a clear pattern in the structure of the secular polynomials as well as in their solutions. One arrives at similar final closed formulae to above. Now one only deals with more variables so that we need two indices to prescribe the complete classification scheme

$$\begin{aligned} & s = s^{(j)} = N + 3 - 4j \\ & r = r^{(j,k)} = t = t^{(j,k)} = -N - 3 + 2j + 2k \\ & k = 1, 2, \ldots, k_{\max}(j) \qquad k_{\max}(j) = N + 2 - 2j \\ & j = 1, 2, \ldots, j_{\max} \qquad j_{\max} = \text{entier}\left[\frac{N+1}{2}\right]. \end{aligned} \tag{42}$$

We may re-emphasize that all the real roots share the symmetry $r = t$ but admit now a different second root s. The physical meaning of these roots is obvious. Thus, the energies of the oscillations in the polynomial well

$$V^{(q=3,k=1)}(r) = ar^2 + br^4 + \cdots + gr^{14}$$

will be proportional to the roots $r^{(j,k)}$. After the change of variables, the roots $s^{(j)}$ will represent energies for the alternative, ‘charged’ polynomial potentials

$$V^{(q=3,k=2)}(r) = \frac{\mathrm{e}}{r} + ar + br^2 + \cdots + fr^6$$

etc [15].

5. Outlook: QES solutions at $q \geqslant 4$ and selected N

5.1. An apparent loss of simplicity at $q = 4$ and $N \leqslant 6$

In our present formulation of the problem (31), we denote the descending diagonals as s_m with $m = 1, 2, 3, 4$ and get the equation

$$\begin{pmatrix} s_1 & 1 & & \\ s_2 & \ddots & \ddots & \\ s_3 & \ddots & \ddots & N-1 \\ s_4 & \ddots & \ddots & s_1 \\ N-1 & \ddots & \ddots & s_2 \\ & \ddots & \ddots & s_3 \\ & & 1 & s_4 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{N-1} \end{pmatrix} = 0. \qquad (43)$$

Its systematic solution does not parallel completely the above-described procedures. In fact, the reduction of the problem to the search for the roots of a single polynomial secular equation $P(x) = 0$ (in the selected auxiliary variable $x = -s_4$) enables us only to factorize $P(x)$ on an extension of the domain of integers,

$$\begin{aligned} P(x) = {} & (x+3)(2x+1-\sqrt{5})(2x+1+\sqrt{5})(2x^2-3x+3\sqrt{5}x+18)(2x^2-3x-3\sqrt{5}x+18) \\ & \times (2x^2-3x-\sqrt{5}x+8+2\sqrt{5})(2x^2-3x+\sqrt{5}x+8-2\sqrt{5}) \\ & \times (x^2+x+\sqrt{5}x+4+\sqrt{5})(x^2+x-\sqrt{5}x+4-\sqrt{5}) \\ & \times (-2\sqrt{5}+8-3x+3\sqrt{5}x+2x^2)(2\sqrt{5}+8-3x-3\sqrt{5}x+2x^2) \\ & \times (-2\sqrt{5}+8+7x-\sqrt{5}x+2x^2)(2\sqrt{5}+8+7x+\sqrt{5}x+2x^2) \\ & \times (2x^2+2x+3-\sqrt{5})(2x^2+2x+3+\sqrt{5}) \\ & \times (\sqrt{5}+3-3x-\sqrt{5}x+2x^2)(-\sqrt{5}+3-3x+\sqrt{5}x+2x^2) \\ & \times (2\sqrt{5}+8-3x+\sqrt{5}x+2x^2)(-2\sqrt{5}+8-3x-\sqrt{5}x+2x^2). \end{aligned}$$

From this lengthy formula it follows that we get

$$s_4^{(1)} = 3 \qquad s_4^{(2)} = \frac{\sqrt{5}+1}{2} \approx 1.618 \qquad s_4^{(3)} = \frac{\sqrt{5}-1}{2} \approx -0.618.$$

There only exist these three real roots s_4 in this case.

A similar computerized procedure also gave us the real roots at $N = 5$. Their inspection leads to the conclusion that $s_2 = s_3$, $s_1 = s_4$. We have not succeeded in an application of our algorithms beyond $N = 5$ yet. Even the $N = 5$ version of equation (43) in its reduction to the condition

$$\begin{aligned} & x^{70} - 936x^{65} + 67\,116x^{60} - 95\,924\,361x^{55} - 74\,979\,131\,949x^{50} + 8568\,894\,879\,002x^{45} \\ & \quad - \cdots - 17\,459\,472\,274\,501\,870\,222\,336x^5 + 142\,630\,535\,951\,654\,322\,176 = 0 \end{aligned}$$

of the vanishing auxiliary polynomial required a fairly long computation for its (still closed and compact) symbolic-manipulation factorization summarized in table 1.

Marginally, it is worth noticing that the choice of $q = 4$ is the first instance where the popular cubic polynomial forces may emerge via the change of variables of [15]. For this reason, in particular, the incomplete character of our $q = 4$ solution might prove challenging

Table 1. Sample of the real roots of equation (43) ($q = 4$).

s_3	s_4
$N = 4$	
3	3
$(\sqrt{5}+1)/2$	$(-\sqrt{5}+1)/2$
$(-\sqrt{5}+1)/2$	$(\sqrt{5}+1)/2$
$N = 5$	
-1	-1
4	4
$\sqrt{5}-1$	$-\sqrt{5}-1$
$-\sqrt{5}-1$	$\sqrt{5}-1$
$(\sqrt{5}+3)/2$	$(-\sqrt{5}+3)/2$
$(-\sqrt{5}+3)/2$	$(\sqrt{5}+3)/2$
$N = 6$	
0	0
5	5
$\sqrt{5}$	$-\sqrt{5}$
$-\sqrt{5}$	$\sqrt{5}$
$(\sqrt{5}+5)/2$	$(-\sqrt{5}+5)/2$
$(-\sqrt{5}+5)/2$	$(\sqrt{5}+5)/2$

in the context of the so-called PT symmetric quantum mechanics where the study of cubic oscillators happened to play a particularly significant role [22].

5.2. *Simplicity re-gained at $q = 5$*

5.2.1. $N = 6$. At $q = 5$ and $N = 6$, the symbolic manipulations using the Gröbner bases [23] generate the secular polynomial in $x = s_5$ which has the slightly deterring form

$$x^{91} - 16\,120x^{85} + 49\,490\,694x^{79} - 286\,066\,906\,320x^{73} - 3553\,475\,147\,614\,293x^{67} \\ - \cdots - 319\,213\,100\,611\,990\,814\,833\,843\,025\,405\,983\,064\,064\,000\,000x = 0.$$

Fortunately, it proves proportional to the polynomial with the mere equidistant and simple real zeros,

$$P_1^{(6)}(x) = x(x^2-1)(x^2-2^2)(x^2-3^2)(x^2-4^2)(x^2-5^2).$$

The rest of the secular polynomial is a product of the other two elementary and positive definite polynomial factors

$$P_2^{(6)}(x) = \prod_{k=1}^{2}(x^2-3kx+3k^2)(x^2+3k^2)(x^2+3kx+3k^2)$$

and

$$P_3^{(6)} = \prod_{k=1}^{5}(x^2-kx+k^2)(x^2+kx+k^2)$$

with another positive definite polynomial

$$P_4^{(6)} = \prod_{k=1}^{12}(x^2-b_kx+c_k)(x^2+b_kx+c_k)$$

Table 2. Coefficients b_k and c_k for $q = 5$ and $N = 6$.

k	c_k	b_k
$1-3$	7	1, 4, 5
$4-6$	13	2, 5, 7
$7-9$	19	1, 7, 8
$10-12$	21	3, 6, 9

Table 3. Real roots of equation (31) at $q = 5$ and $N = 6$.

s_3	s_4	s_5
−5	5	−5
−3	5	−3
−1	5	−1
1	5	1
3	5	3
5	5	5
−5	−1	1
−3	−1	3
−1	−1	−1
1	−1	1
3	−1	−3
5	−1	−1
−5	2	−2
−3	2	0
−1	2	−4
−1	2	2
1	2	−2
1	2	4
3	2	0
5	2	2

where the structure of the two series of coefficients (see their list in table 2) is entirely enigmatic.

The subsequent symbolic manipulations reveal a symmetry $s_2 = s_4$ and $s_1 = s_5$ of all the real eigenvalues. In the pattern summarized in table 3, we recognize a clear indication of a return to the transparency of the $q \leqslant 3$ results which may be written and manipulated in integer arithmetics.

5.2.2. $N = 7$. One should note that in spite of its utterly transparent form, the latter result required a fairly long computing time for its derivation. One encounters new technical challenges here, which will require a more appropriate treatment in the future [24]. Indeed, the comparison of the $N = 6$ secular polynomial equation with its immediate $N = 7$ descendant

$$x^{127} - 60\,071x^{121} + 1021\,190\,617x^{115} - 11\,387\,407\,144\,495x^{109} - \cdots + cx10^6 = 0$$

shows that the last coefficient

$$c = 125\,371\,220\,122\,726\,667\,620\,073\,789\,326\,658\,415\,654\,595\,883\,041\,274\,311\,330\,630\,729\,728$$

now fills almost the whole line. This case failed to be tractable by our current computer code and offers the best illustration of the quick growth of the complexity of the $q \geqslant 5$ constructions with the growth of the QES dimension parameter N.

Table 4. Additional coefficients f_k and g_k at $q = 5$ and $N = 7$.

k	f_k	g_k
$1-3$	28	2, 8, 10
$4-6$	31	4, 7, 11

Fortunately, we are still able to keep the trace of the pattern outlined in tables 2 and 3. Indeed, our new secular polynomial factorizes again in the product of four factors $P_j(x)$, $j = 1, 2, 3, 4$ where only the first one has real zeros,

$$P_1^{(7)}(x) = P_1^{(6)}(x) \cdot (x^2 - 6^2).$$

The further three factors fit the structure of their respective predecessors very well,

$$P_2^{(7)}(x) = P_2^{(6)}(x) \cdot (x^2 - 9x + 27)(x^2 + 27)(x^2 + 9x + 27)$$

and

$$P_3^{(7)} = P_3^{(6)} \cdot (x^2 - 6x + 36)(x^2 + 6x + 36)$$

while

$$P_4^{(7)} = P_4^{(6)} \cdot \prod_{k=1}^{6}(x^2 - f_k x + g_k)(x^2 + f_k x + g_k).$$

The subscript dependence of the new coefficients is listed in table 4.

On the basis of the above factorization we may deduce that the pattern of table 2 survives, *mutatis mutandis*, also the transition to $N = 7$. Indeed, by inspection of tables 2 and 4 one easily proves that the product function $P_2(x)P_3(x)P_4(x)$ has no real zeros and remains positive on the whole real line of x again. A full parallel with the $N = 6$ pattern is achieved and might be conjectured, on this background, for all N, therefore.

6. Summary

Our paper offered new closed solutions of the Schrödinger equation with polynomial potentials at the large angular momenta $\ell \gg 1$. This type of construction proves well founded and motivated, say, in nuclear physics where, quite naturally, the variational calculations in a hyperspherical basis lead to very large values of $\ell = \mathcal{O}(10^3)$ [25]. In such a setting, of course, practically *any* version of the popular $1/\ell$ perturbation expansion (a compact review may be found, e.g., in papers [26]) must necessarily lead to a satisfactory numerical performance.

Our present project was more ambitious. We imagined that a rarely mentioned weak point of all the above perturbative philosophy lies in the notoriously narrow menu of the necessary zero-order approximants H_0 [27]. Indeed, in spite of an amazing universality of all the different $1/\ell$ (better known as $1/N$) expansion techniques (cf, a small sample of the relevant computational tricks in [28]), one usually finds and returns to the common harmonic oscillator $H_0^{(\mathrm{HO})} \equiv H^{(q=0)}$, in spite of the wealth and variability of the underlying physics [29]. For this reason, we recently started to study some alternative possibilities offered by the QES models [17, 20]. In our present continuation of this effort, a decisive extension of the results of this type is given.

Our text reveals the existence and describes the construction of certain fairly large multiplets of 'exceptional' $\ell \gg 1$ bound states for a very broad class of polynomial oscillators. We believe that they might find an immediate application in some phenomenological $D \gg 1$

models where the enhancement of the flexibility of the models with $q > 1$ might lead, say, to a more precise fit of the vibrational spectra, etc.

From the mathematical point of view, the most innovative and characteristic feature of our new $D \gg 1$ QES multiplets lies in the existence of the new *closed and compact* formulae for the QES energies and/or couplings *at all N*. For this reason, the corresponding partially solvable polynomial oscillator Hamiltonians $H_0^{(q,N)}$ might even be understood as lying somewhere in between the QES and ES classes.

Due to such an exceptional transparency of our constructions of $H_0^{(q,N)}$, a facilitated return to the 'more realistic' finite spatial dimensions $D = \mathcal{O}(1)$ might prove tractable by perturbation techniques. Two reasons may be given in favour of such a strategy. First, due to the specific character of our present 'unperturbed' spectra *and* eigenvectors, the perturbation algorithm might be implemented *in integer arithmetics* (i.e., without rounding errors) in a way outlined, preliminarily, in [17] at $q = 1$. Second, the evaluation of the few lowest orders might suffice. This expectation follows from the enhanced flexibility of the available zero-order Hamiltonians. *A priori*, a better convergence of the corrections will be achieved via a better guarantee of 'sufficient smallness' of the difference between the realistic Hamiltonian H and its available approximant H_0.

Of course, the detailed practical implementation of the perturbation technique represents an independent task which must be deferred to a separate publication. With encouraging results, the first steps in this direction have already been performed at $q = 2$ [20]. In parallel, it seems feasible to enlarge further the range of q in zero order. Although one has to deal with fairly complicated symbolic manipulations on the computer beyond $q = 3$, we still intend to perform a deeper analysis of the problems with $q \geqslant 4$ in the nearest future [24].

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